Tutorial 5 (Mar 3, 5)

Leon Li

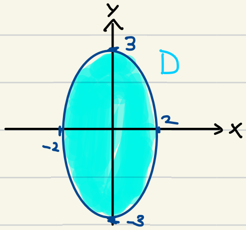
ylli@math.cuhk.edu.hk

Q1) Evaluate $\iint_D x^2 \, dA$, where D is the region bounded by the ellipse $9x^2+4y^2=36$.

Sol) Idea: Compute the integral via a suitable change of variables.

Step 1: Describe D.

Picture:



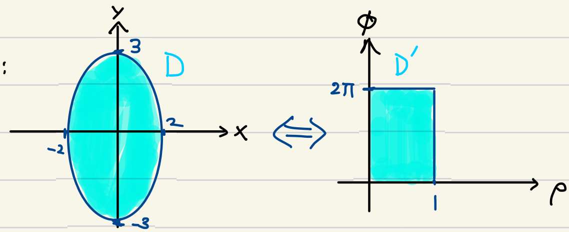
$$D = \{(x,y) \in \mathbb{R}^2 \mid 9x^2+4y^2 \leq 36\}$$

Step 2: Apply a change of variable $\begin{cases} x = 2\rho\cos\phi \\ y = 3\rho\sin\phi. \end{cases}$, where $\rho \geq 0$, $0 \leq \phi < 2\pi$

then $9x^2+4y^2 \leq 36 \iff 36\rho^2(\cos^2\phi+\sin^2\phi) \leq 36 \iff \rho \leq 1$

$$D' = \{(\rho,\phi) \mid 0 \leq \rho \leq 1 ; 0 \leq \phi \leq 2\pi\}$$

Picture:



Step 3: Compute the Jacobian $\frac{\partial(x,y)}{\partial(\rho,\phi)}$.

$$\frac{\partial(x,y)}{\partial(\rho,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} 2\cos\phi & -2\rho\sin\phi \\ 3\sin\phi & 3\rho\cos\phi \end{vmatrix} = 6\rho(\cos^2\phi+\sin^2\phi) = 6\rho$$

Step 4: Apply the change of variables formula.

$$\iint_D x^2 \, dx\,dy = \iint_{D'} (2\rho\cos\phi)^2 (|6\rho| \, d\rho \, d\phi) = 24\int_0^{2\pi}\int_0^1 \rho^3\cos^2\phi \, d\rho \, d\phi$$

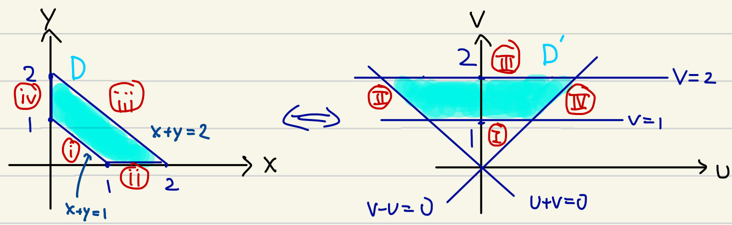
$$= 24\left(\int_0^{2\pi}\cos^2\phi \, d\phi\right)\left(\int_0^1 \rho^3 \, d\rho\right) = 24\left[\frac{\phi}{2}+\frac{\sin 2\phi}{4}\right]_0^{2\pi} \cdot \left[\frac{\rho^4}{4}\right]_0^1 = 24 \cdot \pi \cdot \frac{1}{4} = 6\pi$$

Q2) Evaluate $\iint_D \cos\frac{y-x}{y+x}\,dA$, where D is the trapezoidal region with vertices $(1,0), (2,0), (0,2), (0,1)$.

Sol) Idea: Simplify the integrand by a change of variables and compute the integral.

Step 1: Apply a change of variables $\begin{cases} y-x=u \\ y+x=v \end{cases} \Leftrightarrow \begin{cases} x=\frac{v-u}{2} \\ y=\frac{v+u}{2} \end{cases}$

Picture:



Boundary equations: $\begin{Bmatrix} (i) & x+y=1 \\ (ii) & y=0 \\ (iii) & x+y=2 \\ (iv) & x=0 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} (I) & v=1 \\ (II) & u+v=0 \\ (III) & v=2 \\ (IV) & v-u=0 \end{Bmatrix}$ $\therefore D'=\{(u,v)\in\mathbb{R}^2 \mid 1\le v\le 2, -v\le u\le v\}$

Step 2: Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Step 3: Apply the change of variables formula.

$$\iint_D \cos\left(\frac{y-x}{y+x}\right)dx\,dy = \iint_{D'} \cos\frac{u}{v}\cdot\left(\left|-\tfrac{1}{2}\right|du\,dv\right) = \frac{1}{2}\int_1^2\int_{-v}^{v}\cos\frac{u}{v}\,du\,dv = \frac{1}{2}\int_1^2\left[v\sin\frac{u}{v}\right]_{-v}^{v}dv$$

$$= \frac{1}{2}\int_1^2 (v\sin 1-(-v\sin 1))\,dv = (\sin 1)\cdot\int_1^2 v\,dv = \sin 1\cdot\left[\frac{v^2}{2}\right]_1^2 = \frac{3}{2}\sin 1$$

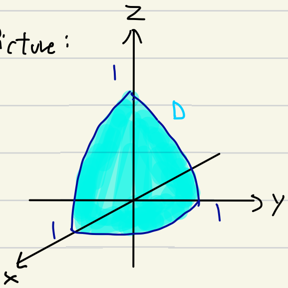
Q3) Evaluate $\iiint_D x e^{x^2+y^2+z^2} dV$, where D is the portion of the unit ball $x^2+y^2+z^2 \leq 1$ that lies in the first octant.

Sol) Idea: Adopt the method of change of variables using spherical coordinates.

Step 1: Describe D.

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x,y,z \geq 0 \\ x^2+y^2+z^2 \leq 1 \end{array} \right\}$$

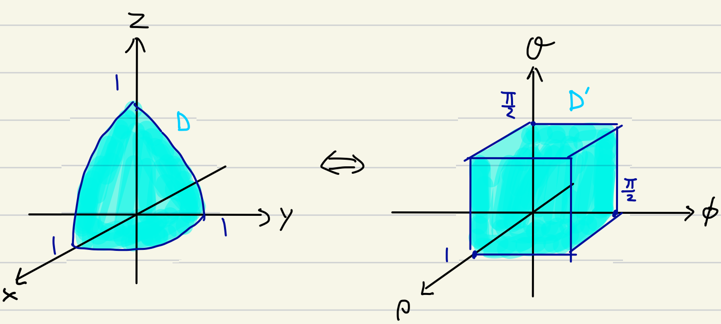
Picture:



Step 2: Apply a change of variables $\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$, where $\begin{cases} \rho \geq 0 \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta < 2\pi \end{cases}$

then $\begin{cases} x^2+y^2+z^2 \leq 1 \Leftrightarrow \rho \leq 1 \\ z \geq 0 \Leftrightarrow \phi \leq \frac{\pi}{2} \\ x,y \geq 0 \Leftrightarrow \theta \leq \frac{\pi}{2} \end{cases}$ $\therefore D' = \left\{ (\rho,\phi,\theta) \;\middle|\; \begin{array}{l} 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \frac{\pi}{2} \\ 0 \leq \theta \leq \frac{\pi}{2} \end{array} \right\}$

Picture:



Step 3: Compute the Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}$.

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi \begin{vmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2\sin\phi\left((-\sin\theta)(-\sin^2\phi\sin\theta - \cos^2\phi\sin\theta) - \cos\theta(-\sin^2\phi\cos\theta - \cos^2\phi\cos\theta)\right)$$

$$= \rho^2\sin\phi(\sin^2\theta + \cos^2\theta) = \rho^2\sin\phi.$$

Step 4: Apply the change of variable formula.

$$\iiint_D x e^{x^2+y^2+z^2}\,dx\,dy\,dz = \iiint_{D'} (\rho\sin\phi\cos\theta)\, e^{\rho^2}\left(|\rho^2\sin\phi|\,d\rho\,d\phi\,d\theta\right)$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3 e^{\rho^2}\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta$$

$$= \left(\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\right)\left(\int_0^{\frac{\pi}{2}}\sin^2\phi\,d\phi\right)\left(\int_0^1 \rho^3 e^{\rho^2}\,d\rho\right)$$

$$= \left[\sin\theta\right]_0^{\frac{\pi}{2}} \cdot \left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{\frac{\pi}{2}} \cdot \left(\left[\frac{\rho^2 e^{\rho^2}}{2}\right]_0^1 - \int_0^1 e^{\rho^2}\cdot d(\rho^2)\right)$$

$$= 1\cdot\frac{\pi}{4}\cdot\frac{1}{2}\left(e - \left[e^{\rho^2}\right]_0^1\right) = \frac{\pi}{8}$$

Remark: Evaluating the triple integral using spherical coordinates is the same as step 4.